PAUTA PRIMERA PEP

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Pregunta 1

Utilice Serie de Fourier para demostrar la identidad trigonométrica

$$\sin^3(x) = \tfrac{3}{4}\sin(x) - \tfrac{1}{4}\sin(3x)$$

Solución:

Se calcula la Serie de Fourier de $f(x) = \sin^3(x)$ en $[-\pi, \pi]$ .Como f es impar, la serie será:

$$\sum_{n=1}^{\infty} b_n \sin nx \quad ; \quad \text{con} \quad b_n = \tfrac{2}{\pi}\int_0^{\pi} \sin^3 x \sin nx dx$$

$$\int_0^{\pi} \sin^3 x \sin nx dx = -\sin^3 x \tfrac{\cos nx}{n} \quad /_o^{\pi} + \tfrac{3}{n}\int_o^{\pi} \cos nx \sin^2 x \cos x dx$$

$$= \tfrac{3}{n}\int_o^{\pi} \sin^2 x \tfrac{1}{2}\left[\cos(n+1)x + \cos(n-1)x\right] dx \tag{1}$$

Para $n = 1$

$$b_1 = \tfrac{2}{\pi}\tfrac{3}{1}\int_o^{\pi} \cos^2 x \sin^2 x dx = \tfrac{2\cdot 3}{\pi}\int_o^{\pi} \tfrac{\sin^2 2x}{4} dx = \tfrac{2\cdot 3}{\pi \cdot 4}\int_o^{\pi} \tfrac{1-\cos 4x}{2} dx$$

$$b_1 = \tfrac{2\cdot 3}{\pi\cdot 4}\tfrac{\pi}{2} = \tfrac{3}{4}$$

Para $n > 1$ en (1)

$$= \tfrac{3}{2n}\left[\sin^2 x\left(\tfrac{\sin(n+1)x}{n+1} + \tfrac{\sin(n-1)x}{n-1}\right)/_o^{\pi} - \int_o^{\pi}\left(\tfrac{\sin(n+1)x}{n+1} + \tfrac{\sin(n-1)x}{n-1}\right)\sin 2x dx\right]$$

$$= -\tfrac{3}{2n}\int_o^{\pi}\left(\tfrac{\sin(n+1)x}{n+1} + \tfrac{\sin(n-1)x}{n-1}\right)\sin 2x dx$$

$$= -\tfrac{3}{2n}\tfrac{1}{n+1}\tfrac{1}{2}\int_o^{\pi}(\cos(n-1)x - \cos(n+3)x)dx - \tfrac{3}{2n}\tfrac{1}{n-1}\tfrac{1}{2}\int_o^{\pi}(\cos(n-3)x - \cos(n+1)x)dx$$

$$= 0 \qquad \forall\ n \neq 3$$

Para $n = 3$ el cálculo directo:

$$b_3 = -\tfrac{3}{2\cdot 3\cdot 2}\tfrac{\pi}{2}\tfrac{2}{\pi} = -\tfrac{1}{4}$$

Luego por teorema de convergencia dada la continuidad de f se tiene:

$\sin^3(x) = \frac{3}{4}\sin(x) - \frac{1}{4}\sin(3x)$

Pregunta 2

Dada la curva $\overrightarrow{r}(t) = (\cos t, \sin t, 1 - \cos t) \quad$ con $0 \leqslant t \leqslant 2\pi$

a) En el punto $(0, -1, 1)$ de la curva descrita por la trayectoria $\overrightarrow{r}(t)$, determine la ecuación del plano Osculador.

b)En el instante $t_0 = \frac{3}{2}\pi$ la partícula se escapa por la tangente.¿En qué instante impacta al plano XY si se considera que sale de $\overrightarrow{r}(0)$?

c)Calcular la curvatura K(t). Determinar los instantes en que la curvatura es máxima y en los que esta es mínima en $[0, 2\pi]$

Solución:

a)$\overrightarrow{r}(t) = (\cos t, \sin t, 1 - \cos t) \Longrightarrow$
$\overrightarrow{r}'(t) = (-\sin t, \cos t, \sin t)$ y $\overrightarrow{r}'(\frac{3}{2}\pi) = (1, 0, -1)$

$\overrightarrow{r}''(t)(-\cos t, -\sin t, \cos t)$ y $\overrightarrow{r}''(\frac{3}{2}\pi) = (0, 1, 0)$

$$\overrightarrow{r}'(\tfrac{3}{2}\pi) \times \overrightarrow{r}''(\tfrac{3}{2}\pi) = \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix} = \widehat{i} + \widehat{k}$$

$\therefore \widehat{B} = \frac{1}{\sqrt{2}}(1, 0, -1)$

Ecuación del plano Osculador:
$\frac{1}{\sqrt{2}}(x - 0) + 0(y + 1) + \frac{1}{\sqrt{2}}(z - 1) = 0$
$\Longrightarrow x + z = 1$

b) Como $\overrightarrow{r}'(\frac{3}{2}\pi) = (1, 0, -1) \Longrightarrow \widehat{\mathbf{T}} = \frac{1}{\sqrt{2}}(1, 0, -1)$
$p(t) = (0, -1, 1) + t\frac{1}{\sqrt{2}}(1, 0, -1);\quad$ ecuación de la recta tangente,
Impacta cuando $z(t) = 1 - t\frac{1}{\sqrt{2}} = 0 \Longrightarrow t = \sqrt{2}$
Instante de impacto: $\frac{3}{2}\pi + \sqrt{2}$

$$\text{c)}\overrightarrow{r}'(t) \times \overrightarrow{r}''(t) = \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ -\sin t & \cos t & \sin t \\ -\cos t & -\sin t & \cos t \end{vmatrix} = \widehat{i} + 0\widehat{j} + \widehat{k}$$

$K(t) = \frac{\left\|\overrightarrow{r}'(t) \times \overrightarrow{r}''(t)\right\|}{\left\|\overrightarrow{r}'(t)\right\|^3} \Longrightarrow K(t) = \frac{\sqrt{2}}{(1+\sin^2 t)^{\frac{3}{2}}}$

Derivamos para obtener valores extremos:

$K`(t) = -\frac{3\sqrt{2}\sin t \cos t}{(1+\sin^2 t)^{\frac{5}{2}}}$
$K`(t) = 0 \Longleftrightarrow \sin t = 0 \quad \text{y} \quad \cos t = 0$
En $t \in [0, 2\pi]$ los puntos críticos son: $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

$K"(t) = -\frac{3}{\sqrt{2}} \frac{2\cos 2t(1+\sin^2 t) - \sin^2 2t}{(1+\sin^2 t)^4}$

$K"(0) = K"(\pi) = K"(2\pi) = -3\sqrt{2} \ < 0$
$K"(\frac{\pi}{2}) = K"(\frac{3\pi}{2}) = 6\sqrt{2} \quad > 0$

$\therefore K$ max para $t = 0$, $t = \pi$ y $t = 2\pi$
K min para $t = \frac{\pi}{2}, t = \frac{3\pi}{2}$

Pregunta 3
Una partícula se mueve a lo largo de la espiral $\rho = e^{\theta}$. Considere la parametrización $c(\theta) = (x(\theta), y(\theta))$ donde $x(\theta) = e^{\theta}\cos\theta, \quad y(\theta) = e^{\theta}\sin\theta$.

a)Verifique que $\theta(t) = \ln(\frac{5}{\sqrt{2}}t), t > 0$ permite obtener la reparametrización

$$\widetilde{c}(t) = \frac{5}{\sqrt{2}}t\left((\cos(\ln(\frac{5}{\sqrt{2}}t)), \sin(\ln(\frac{5}{\sqrt{2}}t))\right)$$

Pruebe además que si la partícula se dezplaza según $\widetilde{c}$,su rapidez es constante.

b)Obtenga la velocidad y aceleración de la partícula cuando esta se dezplaza según $\widetilde{c}$, en el punto $\theta = \frac{\pi}{4}$.

Solución:

a) $c(\theta) = (e^{\theta}\cos\theta, e^{\theta}\sin\theta) \Longrightarrow \widetilde{c}(t) = e^{\theta(t)}(\cos(\theta(t)), \sin(\theta(t)))$
$\widetilde{c}(t) = e^{\ln(\frac{5}{\sqrt{2}}t)}(\cos(\ln(\frac{5}{\sqrt{2}}t)), \sin(\ln(\frac{5}{\sqrt{2}}t)))$

$\widetilde{c}(t) = \frac{5}{\sqrt{2}}t(\cos(\ln(\frac{5}{\sqrt{2}}t)), \sin(\ln(\frac{5}{\sqrt{2}}t)))$
$\widetilde{c}(t) = \frac{5}{\sqrt{2}}(\cos(\ln(\frac{5}{\sqrt{2}}t)) - \sin(\ln(\frac{5}{\sqrt{2}}t)), \cos(\ln(\frac{5}{\sqrt{2}}t)) + \sin(\ln(\frac{5}{\sqrt{2}}t)))$
$\|\widetilde{c}(t)\| = \left[\left(\frac{5}{\sqrt{2}}\right)^2 \cdot 2\right]^{\frac{1}{2}} = \left[\frac{5^2}{2} \cdot 2\right]^{\frac{1}{2}} = 5$
$\|\widetilde{c}(t)\| = 5 \qquad \forall \ t > 0$
Luego la rapidez es constante.

b) $\theta = \ln(\frac{5}{\sqrt{2}}t) \Longleftrightarrow t = \frac{\sqrt{2}}{5}\ e^{\frac{\pi}{4}}$
$x(t) = \frac{5}{\sqrt{2}}t\cos(\ln(\frac{5}{\sqrt{2}}t)) \Longrightarrow x\'(t) = \frac{5}{\sqrt{2}}\left(\cos(\ln(\frac{5}{\sqrt{2}}t)) - \sin(\ln(\frac{5}{\sqrt{2}}t))\right)$

$y(t) = \frac{5}{\sqrt{2}} t \sin(\ln(\frac{5}{\sqrt{2}} t)) \Longrightarrow y\acute{}(t) = \frac{5}{\sqrt{2}} \Big(\sin(\ln(\frac{5}{\sqrt{2}} t)) + \cos(\ln(\frac{5}{\sqrt{2}} t)) \Big)$

en $\theta = \frac{\pi}{4}$, es decir $t_0 = \frac{\sqrt{2}}{5} \ e^{\frac{\pi}{4}}$ se tiene:
$\overrightarrow{v} = \tilde{c}(t_0) = \Big(x`(\frac{\sqrt{2}}{5} e^{\frac{\pi}{4}}), y\acute{}(\frac{\sqrt{2}}{5} e^{\frac{\pi}{4}}) \Big) = \frac{5}{\sqrt{2}} \left(\cos(\frac{\pi}{4})) - \sin(\frac{\pi}{4})), \cos(\frac{\pi}{4}) + \sin(\frac{\pi}{4}) \right)$

$\overrightarrow{v} = (0, 5)$

$x"(t) = -\frac{5}{\sqrt{2}t} \Big(\sin(\ln(\frac{5}{\sqrt{2}} t) + \cos(\ln(\frac{5}{\sqrt{2}} t)) \Big)$
$y"(t) = \frac{5}{\sqrt{2}t} \Big(\cos(\ln(\frac{5}{\sqrt{2}} t)) - \sin(\ln(\frac{5}{\sqrt{2}} t) \Big)$

$\overrightarrow{a} = \Big(x"(\frac{\sqrt{2}}{5} e^{\frac{\pi}{4}}), y"(\frac{\sqrt{2}}{5} e^{\frac{\pi}{4}}) \Big)$
$\overrightarrow{a} = \Big(-\frac{5}{\sqrt{2}} \frac{5}{\sqrt{2} e^{\frac{\pi}{4}}} \left(\sin(\frac{\pi}{4}) + \cos(\frac{\pi}{4}) \right), \frac{5}{\sqrt{2}t} \frac{5}{\sqrt{2} e^{\frac{\pi}{4}}} \left(\cos(\frac{\pi}{4}) - \sin(\frac{\pi}{4}) \right) \Big)$

$\overrightarrow{a} = (\frac{-25}{\sqrt{2}} e^{-\frac{\pi}{4}}, 0)$